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On the high-energy behaviour of scattering phase shifts for Coulomb-like potentials

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Abstract. We discuss the high-energy behaviour of scattering phase shifts for a large class of spherically symmetric Coulomb-like potentials. Some results which have been known only for short-range potentials are extended to this larger class. In particular, by iterating appropriate Volterra integral equations we derive an asymptotic expression for the phase shifts, which is valid whenever the short-range part of the potential is integrable. When applying our results to the problem of the interference between Coulomb and short-range interactions we obtain an estimate for the high-energy behaviour of the Coulomb-interference effect in the phase shifts.

1. Introduction

Rigorous results on partial-wave scattering amplitudes for Coulomb-like potentials have been derived by various authors. For example, Cornille and Martin (1962), considering the Coulomb potential $V^{c}(r)=\gamma/r$, $\gamma\in\mathbb{R}$, plus an additional short-range potential $V(r)$ of the type

$$V(r)=\int_{\mu}^{\infty}\mathrm{d}t\,C(t)\exp(-tr),\qquad |C(t)|<\text{constant}\times t^{1-\epsilon},\quad \epsilon>0,$$

discussed the analytic structure of the S-wave scattering amplitude for complex values of the momentum k. These investigations were generalised by Mentkovski (1965) to the class of short-range potentials fulfilling

$$\int_{0}^{\infty}\mathrm{d}r\,r|V(r)|<\infty.$$

He also considered higher partial waves. Subsequently Klarsfeld (1967) studied the analyticity of Coulomb-like partial-wave amplitudes for complex values of the angular momentum l and for positive values of k. He allowed for potentials $V(r)$ which are less singular than r^{-2} at the origin and vanish faster than r^{-1} at infinity.

In the present paper we study the high-energy behaviour of partial-wave scattering amplitudes corresponding to Schrödinger operators h_l, which are defined as the

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Friedrichs extension of the operators (cf § 3)

$$\dot{h}_l = -\frac{d^2}{dr^2} + \frac{l(l+1)+\alpha^2-\frac{1}{4}}{r^2} + \frac{\gamma}{r} + V(r), \qquad r>0, \quad l \in \mathbb{N}_0, \quad \alpha > 0, \quad \gamma \in \mathbb{R},$$
$$D(\dot{h}_l) = \{f \in C_0^\infty(0,\infty) | \dot{h}_l f \in L^2(0,\infty)\}. \tag{1.1}$$

(We use units: $e = \hbar = 2m = 1$.) In deriving our results we only need to assume that the short-range potential $V(r)$ is locally integrable and fulfils the conditions

$$\int_0^R dr\, r |V(r)| < \infty, \qquad \int_R^\infty dr\, |V(r)| < \infty \qquad \text{for some } R > 0. \tag{1.2}$$

In § 2 we give a detailed description of the 'unperturbed' Hamiltonian $h_l^{(0)}$, which is obtained from (1.1) by setting $V(r) = 0$. In particular we discuss its spectral and scattering properties as well as its Green and Jost functions. In § 3 we first describe the spectral and scattering properties of the Hamiltonian h_l and then turn to the high-energy limit of the Jost function $\mathscr{f}_l(k)$ belonging to h_l. Furthermore, we prove an asymptotic expansion for the phase shift $\delta_l(k) = -\arg \mathscr{f}_l(k)$ which exhibits the well-known fact that the high-energy behaviour of $\delta_l(k)$ depends crucially on the behaviour of the potential near the origin. For the special case of integrable $V(r)$ we derive an asymptotic formula for $\delta_l(k)$ which turns out to be a generalisation of an analogous result in the short-range case $\gamma = 0$, $\alpha = \frac{1}{2}$. We add some remarks concerning the problem of the interference between Coulomb and short-range interactions.

2. The unperturbed Hamiltonian $h_l^{(0)}$

In this section we state the spectral and scattering properties of the unperturbed Hamiltonian $h_l^{(0)}$ defined below. After introducing the corresponding regular and irregular solutions, we examine the Green and Jost functions associated with $h_l^{(0)}$.

In the Hilbert space $L^2(0, \infty)$ we introduce the minimal operator

$$\dot{h}_l^{(0)} = -\frac{d^2}{dr^2} + \frac{l(l+1)+\alpha^2-\frac{1}{4}}{r^2} + \frac{\gamma}{r}, \qquad r>0, \quad l \in \mathbb{N}_0, \quad \alpha \geq 0, \quad \gamma \in \mathbb{R},$$
$$D(\dot{h}_l^{(0)}) = C_0^\infty(0,\infty). \tag{2.1}$$

(The term $(\alpha^2 - \frac{1}{4})/r^2$, while being easily carried along, allows us to generalise all results to more than three space dimensions.)

Because $\dot{h}_l^{(0)}$ is not essentially self-adjoint for $l = 0$, $0 \leq \alpha < 1$, we choose its Friedrichs extension in order to obtain a self-adjoint Hamiltonian. (For a discussion of all other self-adjoint extensions compare Rellich (1943/4), Perelomov and Popov (1970), Zorbas (1979) and Gesztesy and Pittner (1981), particularly for the case $\gamma = 0$.) The domain $D(h_l^{(0)})$ of the Friedrichs extension $h_l^{(0)}$ of $\dot{h}_l^{(0)}$ can be characterised by (Combescure and Ginibre 1975, Kalf 1978)

$$D(h_0^{(0)}) = \left\{ f | f' \in A_{\text{loc}}(0,\infty); f, f' - \frac{1}{2r} f, -f'' - \frac{1}{4r^2} f + \frac{\gamma}{r} f \in L^2(0,\infty) \right\}$$
$$\text{if } l = \alpha = 0, \tag{2.2}$$

$$D(h_0^{(0)}) = \left\{f|f' \in A_{\mathrm{loc}}(0,\infty); f(0_+)=0; f, f', -f'' + \frac{\alpha^2 - \frac{1}{4}}{r^2} f + \frac{\gamma}{r} f \in L^2(0,\infty)\right\}$$

$$\text{if } l=0, \quad 0<\alpha<1. \tag{2.3}$$

Here $A_{\mathrm{loc}}(a, b)$ denotes the set of locally absolutely continuous functions on the interval (a, b).

For $l \geqslant 1$, or $l=0$ and $\alpha \geqslant 1$, $\dot{h}_l^{(0)}$ is essentially self-adjoint, and the domain of its self-adjoint closure $h_l^{(0)}$ is given by

$$D(h_l^{(0)}) = \left\{f|f' \in A_{\mathrm{loc}}(0,\infty); f(0_+)=0; f, f', -f'' + \frac{l(l+1)+\alpha^2 - \frac{1}{4}}{r^2} f + \frac{\gamma}{r} f \in L^2(0,\infty)\right\},$$

$$l \geqslant 1, \qquad \text{or } l=0 \quad \text{and} \quad \alpha \geqslant 1. \tag{2.4}$$

(The conditions $f(0_+)=0$, $f' \in L^2(0,\infty)$ could be dropped in (2.4), since they follow already from the other ones. Note that in the case $\alpha > 0$ Hardy's inequality

$$\int_0^\infty \mathrm{d}r \frac{|f(r)|^2}{r^2} \leqslant 4 \int_0^\infty \mathrm{d}r\, |f'(r)|^2, \qquad f(0_+)=0$$

obviously implies $\gamma f/r \in L^2(0,\infty)$ in (2.3) and (2.4).)

Having defined the self-adjoint Hamiltonian $h_l^{(0)}$ for all relevant values of the parameters l, α, γ, we turn to its spectral properties:

Proposition 1. For all $l \in \mathbb{N}_0$, $\alpha \geqslant 0$, $\gamma \in \mathbb{R}$ the spectrum† of $h_l^{(0)}$ is simple, its singular continuous part is empty, and no positive eigenvalues occur. In particular, for $\gamma \geqslant 0$ the spectrum of $h_l^{(0)}$ is purely absolutely continuous,

$$\sigma(h_l^{(0)}) = \sigma_{\mathrm{ac}}(h_l^{(0)}) = [0,\infty), \qquad \gamma \geqslant 0. \tag{2.5}$$

For $\gamma < 0$ the essential spectrum of $h_l^{(0)}$ is purely absolutely continuous,

$$\sigma_{\mathrm{ess}}(h_l^{(0)}) = \sigma_{\mathrm{ac}}(h_l^{(0)}) = [0,\infty), \qquad \gamma < 0, \tag{2.6}$$

and the point spectrum consists of

$$\sigma_{\mathrm{P}}(h_l^{(0)}) = \left\{ -\frac{\gamma^2}{4[n+\frac{1}{2}+(l^2+l+\alpha^2)^{1/2}]^2} \middle| n, l = 0, 1, 2, \ldots \right\}, \qquad \gamma < 0. \tag{2.7}$$

The corresponding eigenfunctions $\psi_{n,l}$ are given by

$$\psi_{n,l}(r) = C_{n,l} r^\lambda \exp\left(-\frac{r|\gamma|}{2(n+\lambda)}\right) {}_1F_1\left(-n; 2\lambda; \frac{r|\gamma|}{n+\lambda}\right), \qquad \gamma<0, \quad n, l = 0, 1, 2, \ldots, \tag{2.8}$$

where

$$\lambda = \tfrac{1}{2} + (l^2 + l + \alpha^2)^{1/2}$$

and ${}_1F_1(a; b; z)$ denotes the regular confluent hypergeometric function (Abramowitz and Stegun 1972).

Proof. The fact that $\sigma(h_l^{(0)})$ is simple and contains no singular continuous part was proved by Weidmann (1967, theorem 5.1). The absence of positive eigenvalues and the

† For the definition of various kinds of spectra occurring in proposition 1 compare Reed and Simon (1972).

absence of any eigenvalue for $\gamma \geqslant 0$ is a simple consequence of the virial theorem as proved in Gesztesy and Pittner (1980). Equations (2.7) and (2.8) can be proved by direct computation.

It is interesting to note that the eigenvalues (2.7) can also be obtained by purely algebraic methods using the non-invariance group SU(1, 1) (Bacry and Richard 1967).

In order to describe the scattering properties of $h_l^{(0)}$ we state:

Proposition 2. For all $l \in \mathbb{N}_0$, $\alpha \geqslant 0$, $\gamma \in \mathbb{R}$ the modified wave operators

$$\Omega_{\pm,l}^{(0)} = \text{s-}\lim_{t\to\pm\infty} \exp(\mathrm{i}th_l^{(0)})U_l^{\mathrm{c}}(t) \tag{2.9}$$

exist and are strongly asymptotically complete,

$$R(\Omega_{\pm,l}^{(0)}) = H_{\mathrm{ac}}(h_l^{(0)}) = R(E_l^{(0)}((0,\infty))). \tag{2.10}$$

Here $E_l^{(0)}(B)$ denotes the spectral projection for $h_l^{(0)}$ corresponding to the set B, and $U_l^{\mathrm{c}}(t)$ is defined by

$$(U_l^{\mathrm{c}}(t)f)(r) = \text{s-}\lim_{R\to\infty}\int_0^R \mathrm{d}k\,\tilde{f}(k)\exp\left(-\mathrm{i}tk^2 - \frac{\mathrm{i}\gamma(\operatorname{sgn} t)\ln(4k^2|t|)}{2k}\right)(kr)^{1/2}J_{l+1/2}(kr),$$

$$\tilde{f}(k) = \text{s-}\lim_{R\to\infty}\int_0^R \mathrm{d}r\,f(r)(kr)^{1/2}J_{l+1/2}(kr), \qquad f(r)\in L^2(0,\infty). \tag{2.11}$$

The phase shifts $\delta_l^{(0)}(k)$ corresponding to the unitary scattering operator

$$S_l^{(0)} = (\Omega_{+,l}^{(0)})^*\Omega_{-,l}^{(0)} \tag{2.12}$$

are given by

$$\begin{aligned}\delta_l^{(0)}(k) &= \arg(\Gamma(\lambda + \mathrm{i}\gamma/2k)) + \pi(l+1-\lambda)/2\\ &= \arg[\Gamma(\tfrac{1}{2} + (l^2+l+\alpha^2)^{1/2} + \mathrm{i}\gamma/2k)] + \pi[l+\tfrac{1}{2} - (l^2+l+\alpha^2)^{1/2}]/2.\end{aligned} \tag{2.13}$$

Proof. The existence of $\Omega_{\pm,l}^{(0)}$ was first proved by Dollard (1964). For a recent proof of asymptotic completeness using geometrical methods compare Enss (1979). Equation (2.13) can be obtained from the eigenfunction expansion of $h_l^{(0)}$ (cf also (2.20)).

Next we turn to regular and irregular solutions of the equation

$$\phi''(r) - \left(\frac{l(l+1)+\alpha^2-\frac{1}{4}}{r^2} + \frac{\gamma}{r} - k^2\right)\phi(r) = 0, \qquad r, k > 0, \quad l \in \mathbb{N}_0, \quad \alpha > 0, \quad \gamma \in \mathbb{R}. \tag{2.14}$$

They are given by:

regular solution

$$\begin{aligned}F_l^{(0)}(k,r) &= A_l(k)(kr)^\lambda \exp(-\mathrm{i}kr)\,{}_1F_1(\lambda - \mathrm{i}\gamma/2k; 2\lambda; 2\mathrm{i}kr),\\ A_l(k) &= 2^{\lambda-1}\exp(-\pi\gamma/4k)|\Gamma(\lambda + \mathrm{i}\gamma/2k)|/\Gamma(2\lambda);\end{aligned} \tag{2.15}$$

irregular solution

$$\begin{aligned}G_l^{(0)}(k,r) &= B_l(k)(kr)^\lambda \exp(-\mathrm{i}kr)U(\lambda - \mathrm{i}\gamma/2k; 2\lambda; 2\mathrm{i}kr) + \mathrm{i}F_l^{(0)}(k,r),\\ B_l(k) &= -\mathrm{i}2^\lambda \exp(\pi\gamma/4k + \mathrm{i}\pi\lambda)\Gamma(\lambda - \mathrm{i}\gamma/2k)/|\Gamma(\lambda + \mathrm{i}\gamma/2k)|;\end{aligned} \tag{2.16}$$

and the Jost solution

$$H_l^{(0)}(k,r)=\exp(-\mathrm{i}\delta_l^{(0)}(k))(G_l^{(0)}(k,r)+\mathrm{i}F_l^{(0)}(k,r)). \tag{2.17}$$

In equation (2.16) $U(a;b;z)$ denotes the irregular confluent hypergeometric function (Abramowitz and Stegun 1972).

The Jost function $\mathscr{f}_l^{(0)}(k)$ is defined by (cf López and Saavedra 1964, Mentkovski 1965, Klarsfeld 1967)

$$\mathscr{f}_l^{(0)}(k)=(1/k)W(H_l^{(0)},F_l^{(0)})=\exp(-\mathrm{i}\delta_l^{(0)}(k)), \tag{2.18}$$

where

$$W(f,g)=f(\partial g/\partial r)-(\partial f/\partial r)g$$

denotes the Wronskian of f and g.

The Green function $g_l^{(0)}(k,r,r')$ corresponding to $h_l^{(0)}$ can be expressed by

$$g_l^{(0)}(k,r,r')=(1/k)(F_l^{(0)}(k,r)G_l^{(0)}(k,r')-F_l^{(0)}(k,r')G_l^{(0)}(k,r)), \qquad k>0. \tag{2.19}$$

By means of the asymptotic behaviour (Abramowitz and Stegun 1972)

$$F_l^{(0)}(k,r)=\begin{cases} A_l(k)(kr)^{\lambda} & \text{as } r\to 0_+ \\ \sin\left(kr-\dfrac{\gamma}{2k}\ln(2kr)-\dfrac{l\pi}{2}+\delta_l^{(0)}(k)\right) & \text{as } r\to\infty, \end{cases} \tag{2.20}$$

$$G_l^{(0)}(k,r)=\begin{cases} \dfrac{-B_0(k)}{\Gamma(\frac{1}{2}-\mathrm{i}\gamma/2k)}(kr)^{1/2}\left(\ln(2\mathrm{i}kr)+\Psi\left(\dfrac{1}{2}-\dfrac{\mathrm{i}\gamma}{2k}\right)\right) & \text{as } r\to 0_+, \quad l=\alpha=0 \\ \dfrac{B_l(k)(2\mathrm{i})^{1-2\lambda}\Gamma(2\lambda-1)}{\Gamma(\lambda-\mathrm{i}\gamma/2k)}(kr)^{1-\lambda} & \text{as } r\to 0_+, \quad \alpha>0 \\ \cos\left(kr-\dfrac{\gamma}{2k}\ln(2kr)-\dfrac{l\pi}{2}+\delta_l^{(0)}(k)\right) & \text{as } r\to\infty \end{cases} \tag{2.21}$$

($\Psi(z)$ denotes Euler's psi function) and

$$|A_l(k)B_l(k)/\Gamma(\lambda-\mathrm{i}\gamma/2k)|=2^{2\lambda-1}/\Gamma(2\lambda),$$

we finally establish the bound

$$|g_l^{(0)}(k,r,r')|\leq\begin{cases} \dfrac{c_l}{k}\left(\dfrac{kr}{1+kr}\right)^{\lambda}\left(\dfrac{kr'}{1+kr'}\right)^{1-\lambda}, & r\geq r', \\ \dfrac{c_l}{k}\left(\dfrac{kr}{1+kr}\right)^{1-\lambda}\left(\dfrac{kr'}{1+kr'}\right)^{\lambda}, & r\leq r', \end{cases} \qquad l\in\mathbb{N}_0, \quad \alpha>0, \quad \gamma\in\mathbb{R}, \tag{2.22}$$

c_l being some appropriate constant.

In order to avoid logarithmic terms like $\ln(2\mathrm{i}kr)$ in the case $\alpha=l=0$ (cf (2.21)), we shall henceforth assume $\alpha>0$.

3. High-energy behaviour of the scattering phase shifts

In this section we analyse the high-energy limit of phase shifts corresponding to

Hamiltonians h_l which are associated with quadratic forms Q_l:

$$Q_l(f, g) = Q_l^{(0)}(f, g) + Q_V(f, g) \text{ on } D(Q_l) = D(Q_l^{(0)}), \qquad l \in \mathbb{N}_0, \quad \alpha > 0, \quad \gamma \in \mathbb{R}; \tag{3.1}$$

here the form $Q_l^{(0)}$ is associated with $h_l^{(0)}$, and the form Q_V is given by

$$Q_V(f, g) = \int_0^\infty \mathrm{d}r\, V(r)\overline{f(r)}g(r), \qquad D(Q_V) = \{h | h, |V|^{1/2} \qquad h \in L^2(0, \infty)\},$$

where the short-range potential $V(r)$ satisfies

$$\int_0^R \mathrm{d}r\, r|V(r)| < \infty, \qquad \int_R^\infty \mathrm{d}r |V(r)| < \infty \qquad \text{for some } R > 0. \tag{3.2}$$

The conditions (3.2) clearly imply that h_l is a semi-bounded self-adjoint operator, since V is infinitesimally form-bounded with respect to $h_l^{(0)}$ (Reed and Simon 1975). In fact one can prove that h_l coincides with the Friedrichs extension of (1.1) (Gesztesy and Pittner 1979). The spectral and scattering properties of h_l are summarised by:

Proposition 3. (*a*) For all $l \in \mathbb{N}_0$, $\alpha > 0$, $\gamma \in \mathbb{R}$ the spectrum of h_l is simple and bounded from below. Its singular continuous part is empty, no positive eigenvalues occur, and the essential spectrum is purely absolutely continuous,

$$\sigma_{\mathrm{ess}}(h_l) = \sigma_{\mathrm{ac}}(h_l) = [0, \infty). \tag{3.3}$$

(*b*) For all $l \in \mathbb{N}_0$, $\alpha > 0$, $\gamma \in \mathbb{R}$ the wave operators

$$\Omega_{\pm,l}(h_l, h_l^{(0)}) = \mathrm{s-}\lim_{t\to\pm\infty} \exp(\mathrm{i}th_l)\exp(-\mathrm{i}th_l^{(0)})E_l^{(0)}((0, \infty)) \tag{3.4}$$

exist and are strongly asymptotically complete,

$$R(\Omega_{\pm,l}(h_l, h_l^{(0)})) = H_{\mathrm{ac}}(h_l) = R(E_l((0, \infty))). \tag{3.5}$$

Here $E_l(B)$ denotes the spectral projection for h_l corresponding to the set B. We also note the connection with stationary scattering theory:

$$\Omega_{\pm,l}(h_l, h_l^{(0)}) = \mathscr{F}_l \exp[\mp\mathrm{i}(\delta_l - \delta_l^{(0)})]\mathscr{F}_{(0),l}^{-1}, \tag{3.6}$$

where the operator $\mathscr{F}_l$ is defined by

$$\mathscr{F}_l: \begin{cases} E_l((0, \infty))L^2(0, \infty) \to L^2(0, \infty) \\ f(r) \to (\mathscr{F}_l f)(k) = \mathrm{s-}\lim\limits_{R\to\infty} \int_0^R \mathrm{d}r\, \psi_l(k, r)f(r), \end{cases} \tag{3.7}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1) + \alpha^2 - \frac{1}{4}}{r^2} + \frac{\gamma}{r} + V(r)\right)\psi_l(k, r) = k^2\psi_l(k, r), \qquad k, r > 0,$$

$$\psi_l(k, r) \underset{r\to\infty}{\overset{k>0}{\longrightarrow}} \left(\frac{2}{\pi}\right)^{1/2} \sin\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_l(k)\right). \tag{3.8}$$

$\mathscr{F}_{(0),l}$ is defined in an analogous way for the case $V(r) = 0$.

Proof. The results of proposition 3(*a*) follow from Weidmann (1967, theorem 5.1). For a proof of proposition 3(*b*) compare Marchesin and O'Carroll (1972).

Next we turn to the regular and irregular solutions of the equation

$$\psi''(r)-\left(\frac{l(l+1)+\alpha^2-\frac{1}{4}}{r^2}+\frac{\gamma}{r}+V(r)-k^2\right)\psi(r)=0;$$

$$k, r>0, \quad l\in\mathbb{N}_0, \quad \alpha>0, \quad \gamma\in\mathbb{R}, \tag{3.9}$$

which may be obtained from the Volterra integral equations

$$F_l(k,r)=F_l^{(0)}(k,r)+\int_0^r \mathrm{d}r'\, g_l^{(0)}(k,r,r')V(r')F_l(k,r') \tag{3.10}$$

and

$$H_l(k,r)=H_l^{(0)}(k,r)-\int_r^\infty \mathrm{d}r'\, g_l^{(0)}(k,r,r')V(r')H_l(k,r') \tag{3.11}$$

respectively. The bound (2.22) for $g_l^{(0)}(k,r,r')$, in exactly the same way as in the short-range case $\gamma=0$, $\alpha=\frac{1}{2}$ (Newton 1966, Amrein *et al* 1977, Chadan and Sabatier 1977), implies existence and uniqueness of the solutions of equations (3.10) and (3.11). Furthermore, equation (2.22) ensures the following bounds, which are again similar to the short-range case:

$$|F_l(k,r)|\geqslant a_l\left(\frac{kr}{1+kr}\right)^\lambda \exp\left(c_l\int_0^r \mathrm{d}r'|V(r')|\frac{r'}{1+kr'}\right)\geqslant a_l(k_0)\left(\frac{kr}{1+kr}\right)^\lambda, \qquad k\geqslant k_0>0,$$

$$a_l(k_0)=a_l\exp\left(c_l\int_0^\infty \mathrm{d}r'|V(r')|\frac{r'}{1+k_0r'}\right), \tag{3.12}$$

$$|F_l(k,r)-F_l^{(0)}(k,r)|\leqslant c_la_l(k_0)\left(\frac{kr}{1+kr}\right)^\lambda\int_0^r \mathrm{d}r'|V(r')|\frac{r'}{1+kr'}, \qquad k\geqslant k_0>0, \tag{3.13}$$

$$|H_l(k,r)|\leqslant b_l\left(\frac{kr}{1+kr}\right)^{1-\lambda}\exp\left(c_l\int_r^\infty \mathrm{d}r'|V(r')|\frac{r'}{1+kr'}\right)\leqslant b_l(k_0)\left(\frac{kr}{1+kr}\right)^{1-\lambda},$$

$$k\geqslant k_0>0 \tag{3.14}$$

$$b_l(k_0)=b_l\exp\left(c_l\int_0^\infty \mathrm{d}r'|V(r')|\frac{r'}{1+k_0r'}\right),$$

and

$$|H_l(k,r)-H_l^{(0)}(k,r)|\leqslant c_lb_l(k_0)\left(\frac{kr}{1+kr}\right)^{1-\lambda}\int_r^\infty \mathrm{d}r'|V(r')|\frac{r'}{1+kr'}, \qquad k\geqslant k_0>0. \tag{3.15}$$

It is evident that analogous bounds hold for $\partial F_l(k,r)/\partial r$ and $\partial H_l(k,r)/\partial r$.

The Jost function $\mathscr{f}_l(k)$ is defined in terms of the Wronskians of H_l and F_l by

$$\mathscr{f}_l(k)=(1/k)W(H_l,F_l), \qquad k>0, \tag{3.16}$$

or equivalently

$$\mathscr{f}_l(k)=\mathscr{f}_l^{(0)}(k)+\frac{1}{k}\int_0^\infty \mathrm{d}r\, V(r)H_l^{(0)}(k,r)F_l(k,r), \qquad k>0. \tag{3.17}$$

After writing

$$\mathscr{f}_l(k)=|\mathscr{f}_l(k)|\exp(-\mathrm{i}\delta_l(k)), \tag{3.18}$$

the relations

$$F_l(k,r) - |\mathscr{f}_l(k)| \sin\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_l(k)\right)\Big|_{r\to\infty} = \mathrm{o}(1)\dagger, \qquad k>0,$$
$$\left|\frac{\partial}{\partial r}F_l(k,r) - k|\mathscr{f}_l(k)| \cos\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_l(k)\right)\right|_{r\to\infty} = \mathrm{o}(1), \qquad k>0 \tag{3.19}$$

follow in the usual manner. Comparison of (3.19) and (3.8) shows that $\delta_l(k) = -\arg \mathscr{f}_l(k)$ actually coincides with $\delta_l(k)$ in proposition 3 up to multiples of π.

After these preliminaries we examine the high-energy limit of $\mathscr{f}_l(k)$. From the asymptotic behaviour of $\mathscr{f}_l^{(0)}(k)$,

$$\mathscr{f}_l^{(0)}(k) \underset{k\to\infty}{=} \exp\left(-\frac{\mathrm{i}\pi(l+1-\lambda)}{2}\right)\left(1 - \frac{\mathrm{i}\gamma}{2k}\Psi(\lambda) + \mathrm{O}(k^{-2})\right), \tag{3.20}$$

and equations (3.12) and (3.17), one immediately finds

$$|\mathscr{f}_l(k) - \mathscr{f}_l^{(0)}(k)| \leqslant a_l(k_0)b_l(k_0)\int_0^\infty \mathrm{d}r|V(r)|\frac{r}{1+kr}, \qquad k \geqslant k_0 > 0, \tag{3.21}$$

and

$$|\mathscr{f}_l(k) - \exp[-\mathrm{i}\pi(l+1-\lambda)/2]| \underset{k\to\infty}{=} \mathrm{o}(1). \tag{3.22}$$

Because of (3.22) we may choose

$$\delta_l(\infty) = \pi(l+1-\lambda)/2 = \pi[l+\tfrac{1}{2} - (l^2+l+\alpha^2)^{1/2}]/2 \tag{3.23}$$

in order to guarantee the uniqueness of $\delta_l(k)$.

The decrease of $\delta_l(k) - \delta_l^{(0)}(k)$ as k tends to infinity clearly depends on the behaviour of $V(r)$ as r tends to zero. This is shown in

Proposition 4. Suppose that

$$\int_0^R \mathrm{d}r\, r^\beta |V(r)| < \infty, \qquad \int_R^\infty \mathrm{d}r|V(r)| < \infty \qquad \text{for some } \beta, R>0, \quad 0<\beta\leqslant 1.$$

Then

$$\int_{k_0}^\infty \mathrm{d}k \frac{|\delta_l(k) - \delta_l^{(0)}(k)|}{k^\beta} < \infty \tag{3.24}$$

and

$$\delta_l(k) - \delta_l^{(0)}(k) \underset{k\to\infty}{=} \mathrm{o}(k^{\beta-1}), \tag{3.25}$$

or equivalently, using (3.20),

$$\delta_l(k) \underset{k\to\infty}{=} \pi[l+\tfrac{1}{2} - (l^2+l+\alpha^2)^{1/2}]/2 + \mathrm{o}(k^{\beta-1}). \tag{3.26}$$

† We use the symbols $f(x) \underset{x\to a}{=} \mathrm{o}(g(x))$ if $\lim_{x\to a} f(x)/g(x) = 0$, and $f(x) \underset{x\to a}{=} \mathrm{O}(g(x))$ if $f(x)/g(x)$ is bounded as x tends to a.

Proof. With the help of (3.12) and (3.17) we infer

$$\int_{k_0}^{\infty} \mathrm{d}k \frac{|\mathrm{Im}[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))]|}{k^{\beta}} \leqslant a_l^2(k_0) \int_0^{\infty} \mathrm{d}r |V(r)| r^{2\lambda} \int_{k_0}^{\infty} \mathrm{d}k \frac{k^{2\lambda-1-\beta}}{(1+kr)^{2\lambda}} < \infty,$$

since

$$r^{2\lambda} \int_{k_0}^{\infty} \mathrm{d}k \frac{k^{2\lambda-1-\beta}}{(1+kr)^{2\lambda}} \underset{r\to\infty}{=} \mathrm{O}(1), \qquad r^{2\lambda} \int_{k_0}^{\infty} \mathrm{d}k \frac{k^{2\lambda-1-\beta}}{(1+kr)^{2\lambda}} \underset{r\to 0_+}{=} \mathrm{O}(r^{\beta}).$$

Because $\mathrm{Im}[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))]$ and $\delta_l(k)-\delta_l^{(0)}(k)$ are of the same order as k tends to infinity, we obtain (3.24).

To prove (3.25) we proceed as follows:

$$\begin{aligned} k^{1-\beta}&|\mathrm{Im}[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))]| \\ &\leqslant a_l^2(k_0) k^{-\beta} \int_0^{\infty} \mathrm{d}r |V(r)| \left(\frac{kr}{1+kr}\right)^{2\lambda} \\ &= a_l^2(k_0) \int_0^R \mathrm{d}r\, r^{\beta} |V(r)| \frac{(kr)^{2\lambda-\beta}}{(1+kr)^{2\lambda}} + a_l^2(k_0) k^{-\beta} \int_R^{\infty} \mathrm{d}r |V(r)| \left(\frac{kr}{1+kr}\right)^{2\lambda}. \end{aligned}$$

From

$$\frac{x^{\delta-\epsilon}}{(1+x)^{\delta}} \leqslant 1 \qquad \text{for } 0 \leqslant \epsilon \leqslant \delta \quad \text{and all } 0 \leqslant x < \infty \tag{3.27}$$

and Lebesgue's dominated convergence theorem, one proves

$$\lim_{k\to\infty} k^{1-\beta} |\mathrm{Im}[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))]| = 0.$$

Of course the case $\beta = 0$, i.e.

$$\int_0^{\infty} \mathrm{d}r |V(r)| < \infty, \tag{3.28}$$

is of particular interest. From

$$|F_l(k, r) - \sin[kr + \pi(1-\lambda)/2]| \underset{k\to\infty}{=} \mathrm{o}(1), \qquad r > 0 \tag{3.29}$$

(which is a consequence of (3.13)), Lebesgue's dominated convergence theorem, and the Riemann–Lebesgue lemma we conclude that

$$\begin{aligned} -\mathrm{Im}&[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))] \\ &= |\not{f}_l(k)| \sin(\delta_l(k) - \delta_l^{(0)}(k)) \\ &= -\frac{1}{k} \int_0^{\infty} \mathrm{d}r\, V(r) F_l^{(0)}(k, r) F_l(k, r) \\ &\underset{k\to\infty}{=} -\frac{1}{k} \int_0^{\infty} \mathrm{d}r\, V(r) \sin^2\left(kr + \frac{\pi(1-\lambda)}{2}\right) + \mathrm{o}(k^{-1}) \\ &\underset{k\to\infty}{=} -\frac{1}{2k} \int_0^{\infty} \mathrm{d}r\, V(r) + \mathrm{o}(k^{-1}). \end{aligned}$$

Thus we have proved:

Proposition 5. Suppose that

$$\int_0^\infty \mathrm{d}r|V(r)| < \infty;$$

then

$$\delta_l(k) - \delta_l^{(0)}(k) \underset{k\to\infty}{=} -\frac{1}{2k}\int_0^\infty \mathrm{d}r\, V(r) + \mathrm{o}(k^{-1}) \tag{3.30}$$

or

$$\delta_l(k) \underset{k\to\infty}{=} \frac{\pi[l+\frac{1}{2}-(l^2+l+\alpha^2)^{1/2}]}{2} + \frac{1}{2k}\left[\gamma\Psi\left(\frac{1}{2}+(l^2+l+\alpha^2)^{1/2}\right) - \int_0^\infty \mathrm{d}r\, V(r)\right] + \mathrm{o}(k^{-1}). \tag{3.31}$$

The latter relation (3.31) is well known in the short-range case $\gamma = 0$, $\alpha = \frac{1}{2}$ (Newton 1966, Amrein *et al* 1977). Note that for $\alpha = \frac{1}{2}$ and l large enough the phase shift $\delta_l(k)$ cannot vanish faster than k^{-1} as $k \to \infty$, irrespective of the shape of $V(r)$ (and of the sign of γ).

Let us now take a look at the problem of the interference between Coulomb and short-range interactions which has aroused considerable interest in the literature recently (cf e.g. Plessas *et al* 1974, Hamilton 1975, Fröhlich *et al* 1978, and references cited therein). We obtain this case by putting $\alpha = \frac{1}{2}$. The Coulomb-interference effect is contained in

$$\hat{\delta}_l^{\mathrm{s}}(k) = \delta_l(k) - \delta_l^{\mathrm{c}}(k), \qquad \delta_l^{\mathrm{c}}(k) = \arg(\Gamma(l+1+\mathrm{i}\gamma/2k)), \tag{3.32}$$

i.e. the phase shift corresponding to the short-range potential acting in the Coulomb field. The above difference is a special case of equations (3.25) and (3.30), namely when $\alpha = \frac{1}{2}$. Now the Coulomb-interference effect is best seen by splitting up $\hat{\delta}_l^{\mathrm{s}}(k)$ further:

$$\hat{\delta}_l^{\mathrm{s}}(k) = \delta_l^{\mathrm{s}}(k) + \Delta_l(k). \tag{3.33}$$

Here $\delta_l^{\mathrm{s}}(k)$ denotes the scattering phase shift corresponding to the short-range potential $V(r)$ alone ($\gamma = 0$, $\alpha = \frac{1}{2}$). The remainder $\Delta_l(k)$ will obviously contain all Coulomb-interference effects. Because equations (3.26) and (3.31) also hold for $\delta_l^{\mathrm{s}}(k)$, we obtain:

Proposition 6. Suppose that

$$\int_0^R \mathrm{d}r\, r^\beta |V(r)| < \infty, \qquad \int_R^\infty \mathrm{d}r|V(r)| < \infty \qquad \text{for some } \beta, R > 0, \quad 0 \leqslant \beta \leqslant 1;$$

then

$$\Delta_l(k) \underset{k\to\infty}{=} \mathrm{o}(k^{\beta-1}). \tag{3.34}$$

4. Highly singular oscillating potentials

The purpose of this section is to show that propositions 3 and 4 (for $\beta = 1$) remain valid if the short-range potential $V(r)$ is singular and oscillating near the origin.

Consider potentials $V(r)$ such that

$$\int_R^\infty \mathrm{d}r|V(r)|<\infty \qquad \text{for all } R>0. \tag{4.1}$$

Define

$$W(r)=-\int_r^\infty \mathrm{d}r'\ V(r') \tag{4.2}$$

and assume

$$\lim_{r\to 0_+} rW(r)=0 \tag{4.3}$$

as well as

$$\int_0^\infty \mathrm{d}r|W(r)|<\infty. \tag{4.4}$$

Standard expressions for $W(r)$ are (Baeteman and Chadan 1975, 1976)

$$W(r)=r^{-\alpha}\sin[\exp(1/r)]\exp(-\mu r), \qquad 0\leqslant\alpha<1, \quad \mu>0$$

or

$$W(r)=r^{-\alpha}(-\ln r)^{-\beta}\theta(\tfrac{1}{2}-r), \qquad 0\leqslant\alpha<1, \qquad \text{or } \alpha=1, \quad \beta>1.$$

These examples clearly show the enlargement of the class of admissible potentials.

It was shown by Baeteman and Chadan (1975, 1976), Combescure and Ginibre (1976) and Chadan and Martin (1977) that in the short-range case $\gamma=0$, $\alpha=\frac{1}{2}$ this class of potentials (the so-called W class) exhibits the same character with respect to Jost functions and spectral and scattering properties as regular potentials satisfying $\int_0^\infty \mathrm{d}r\, r|V(r)|<\infty$. (For recent results on strongly oscillating potentials, cf also Combescure (1979) and Pearson (1979).) We show that this property remains true if an additional Coulomb potential is present. The starting point of this observation is of course an appropriate iteration of (3.10) and (3.11) yielding uniform convergence of the iterated series.

Replacing $V(r)$ by $W'(r)$ and integrating by parts in (3.10), we obtain with the help of (4.3)

$$F_l(k,r)=F_l^{(0)}(k,r)-\int_0^r \mathrm{d}r'\ W(r')\bigg[\bigg(\frac{\partial}{\partial r'}g_l^{(0)}(k,r,r')\bigg)F_l(k,r') + g_l^{(0)}(k,r,r')\bigg(\frac{\partial}{\partial r'}F_l(k,r')\bigg)\bigg], \qquad k>0. \tag{4.5}$$

Differentiation of (4.5) yields

$$\frac{\partial}{\partial r}F_l(k,r)=\frac{\partial}{\partial r}F_l^{(0)}(k,r)+W(r)F_l(k,r) -\int_0^r \mathrm{d}r'\ W(r')\bigg[\bigg(\frac{\partial}{\partial r}\frac{\partial}{\partial r'}g_l^{(0)}(k,r,r')\bigg)F_l(k,r') +\bigg(\frac{\partial}{\partial r}g_l^{(0)}(k,r,r')\bigg)\bigg(\frac{\partial}{\partial r'}F_l(k,r')\bigg)\bigg], \qquad k>0. \tag{4.6}$$

The estimates

$$|g_l^{(0)}(k,r,r')| \leqslant \frac{c_l}{k}\left(\frac{kr}{1+kr}\right)^{\lambda}\left(\frac{kr'}{1+kr'}\right)^{1-\lambda}, \qquad r \geqslant r',$$

$$\left|\frac{\partial}{\partial r'} g_l^{(0)}(k,r,r')\right| \leqslant |\lambda-1| a_l b_l \left(\frac{kr}{1+kr}\right)^{\lambda}\left(\frac{kr'}{1+kr'}\right)^{-\lambda}, \qquad r \geqslant r',$$

$$\left|\frac{\partial}{\partial r} g_l^{(0)}(k,r,r')\right| \leqslant \lambda a_l b_l \left(\frac{kr}{1+kr}\right)^{\lambda-1}\left(\frac{kr'}{1+kr'}\right)^{1-\lambda}, \qquad r \geqslant r',$$

$$\left|\frac{\partial}{\partial r}\frac{\partial}{\partial r'} g_l^{(0)}(k,r,r')\right| \leqslant k\lambda|\lambda-1| a_l b_l \left(\frac{kr}{1+kr}\right)^{\lambda-1}\left(\frac{kr'}{1+kr'}\right)^{-\lambda}, \qquad r \geqslant r', \tag{4.7}$$

$$|F_l^{(0)}(k,r)| \leqslant a_l\left(\frac{kr}{1+kr}\right)^{\lambda}, \qquad \left|\frac{\partial}{\partial r} F_l^{(0)}(k,r)\right| \leqslant k\lambda a_l\left(\frac{kr}{1+kr}\right)^{\lambda-1}, \; c_l \leqslant a_l b_l \tag{4.8}$$

and an iteration of the system (4.5) and (4.6) then show

$$|F_l(k,r)| \leqslant a_l\left(\frac{kr}{1+kr}\right)^{\lambda} \exp\left(a_l b_l(2\lambda-1+c)\int_0^r \mathrm{d}r'|W(r')|\right), \qquad k>0,$$

$$\left|\frac{\partial}{\partial r}F_l(k,r)\right| \leqslant k a_l(\lambda+c)\left(\frac{kr}{1+kr}\right)^{\lambda-1} \exp\left(a_l b_l(2\lambda-1+c)\int_0^r \mathrm{d}r'|W(r')|\right), \qquad k>0, \tag{4.9}$$

where c is defined by

$$c = \max_{r\geqslant 0}\,(r|W(r)|).$$

The bounds (4.9) for the W class are of exactly the same structure as the bounds for potentials satisfying (3.2) (cf (3.12)), and this is obviously true for the bounds of $H_l(k,r)$ and $\partial H_l(k,r)/\partial r$. This observation suggests that there is actually no difference between the class of potentials treated in § 3 and that of singular oscillating potentials obeying (4.1)–(4.4). In particular, proposition 3 remains unchanged, and from

$$\begin{aligned}
&|\mathrm{Im}[\not{f}_l(k)\exp(\mathrm{i}\delta_l^{(0)}(k))]| \\
&\quad \leqslant \frac{1}{k}\int_0^R \mathrm{d}r|W(r)|\left|\left(\frac{\partial}{\partial r}F_l^{(0)}(k,r)\right)F_l(k,r)+F_l^{(0)}(k,r)\left(\frac{\partial}{\partial r}F_l(k,r)\right)\right| \\
&\qquad +\frac{1}{k}|W(R)|\,|F_l^{(0)}(k,R)|\,|F_l(k,R)| \\
&\qquad +\frac{1}{k}\int_R^\infty \mathrm{d}r|V(r)|\,|F_l^{(0)}(k,r)|\,|F_l(k,r)|,
\end{aligned} \tag{4.10}$$

the estimates (4.9), and the fact that one can choose R arbitrarily small one proves

$$\delta_l(k) \underset{k\to\infty}{=} \pi[l+\tfrac{1}{2}-(l^2+l+\alpha^2)^{1/2}]/2+\mathrm{o}(1), \tag{4.11}$$

i.e. the analogue of proposition 4 for $\beta=1$.

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